

# Algebraic Ricci solitons of four-dimensional pseudo-Riemannian generalized symmetric spaces

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## Abstract

We completely classify the algebraic Ricci solitons of four-dimensional pseudo-Riemannian generalized symmetric spaces.

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## 1 Introduction and preliminaries

The concept of the Algebraic Ricci soliton was first introduced by Lauret in Riemannian case ([9]). The definition extends to the pseudo-Riemannian case:

**Definition 1.1.** *Let  $(G, g)$  be a simply connected Lie group equipped with the left-invariant pseudo-Riemannian metric  $g$ , and let  $\mathfrak{g}$  denote the Lie algebra of  $G$ . Then  $g$  is called an algebraic Ricci soliton if it satisfies*

$$\text{Ric} = c\text{Id} + D \tag{1.1}$$

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where  $\text{Ric}$  denotes the Ricci operator,  $c$  is a real number, and  $D \in \text{Der}(\mathfrak{g})$  ( $D$  is a derivation of  $\mathfrak{g}$ ), that is:

$$D[X, Y] = [DX, Y] + [X, DY] \quad \text{for any } X, Y \in \mathfrak{g}. \quad (1.2)$$

In particular, an algebraic Ricci soliton on a solvable Lie group, (a nilpotent Lie group) is called a sol-soliton (a nil-soliton).

Obviously, Einstein metrics on a Lie group are algebraic Ricci solitons.

A Ricci soliton metric  $g$  on a manifold  $M$  is a pseudo-Riemannian metric satisfying

$$\varrho[g] = cg + L_X g, \quad (1.3)$$

where  $L_X$  denotes the Lie derivative in the direction of the vector field  $X$ ,  $\varrho$  is the Ricci tensor and  $c$  is a real constant. A Ricci soliton is said to be shrinking, steady or expanding, if  $\lambda > 0$ ,  $\lambda = 0$  or  $\lambda < 0$ , respectively. Moreover, we say that a Ricci soliton  $(M, g)$  is a gradient Ricci soliton if it admits a vector field  $X$  satisfying  $X = \text{grad}(h)$ , for some potential function  $h$  (see also [4]).

Furthermore, the condition (1.3) is equivalent to  $g(t) = (-2ct + 1)\varphi_{s(t)}^*$  being a solution to the Ricci flow

$$\frac{\partial}{\partial t} g(t)_{ij} = -2\varrho[g(t)]_{ij},$$

where  $\varphi_s$  is the family of diffeomorphisms generated by  $X$  with  $s(t) = \frac{1}{c} \ln(-2ct + 1)$ .

In [9], Lauret studied the relation between sol-solitons and Ricci solitons on Riemannian manifolds. More precisely, he proved that any left-invariant Riemannian sol-soliton metric is a Ricci soliton. This was extended by the second author to the pseudo-Riemannian case :

**Theorem 1.1** ([10]). *Let  $(G, g)$  be a simply connected Lie group endowed with a left-invariant pseudo-Riemannian metric  $g$ . If  $g$  is sol-soliton, then  $g$  is the Ricci soliton, that is,  $g$  satisfy (1.3), such that*

$$X = \left. \frac{d\varphi_t}{dt} \right|_{t=0} (p) \quad \text{and} \quad \exp\left(\frac{t}{2}D\right) = d\varphi_t|_e,$$

where  $e$  denotes the identity element of  $G$ .

Note that the above theorem is correct, if changing sol-soliton into an algebraic Ricci soliton.

On the other hand, if  $(M, g)$  is a homogeneous (pseudo-)Riemannian manifold, then there exists a group  $G$  of isometries acting transitively on it [11]. Such  $(M, g)$  can be then identified with  $(G/H, g)$ , where  $H$  is the isotropy group at a fixed point  $p$  of  $M$ . Let  $\mathfrak{g}$  denote the Lie algebra of  $G$  and fix an  $\text{Ad}(H)$ -invariant decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , where  $\mathfrak{h}$  is the Lie algebra of  $H$ . The space  $\mathfrak{m}$  is naturally identified with  $T_p M$ . It must be noted, that such decomposition exists always in the Riemannian case since homogeneous Riemannian manifolds

are reductive. However, in the pseudo-Riemannian setting the homogeneous pseudo-Riemannian manifold need to be reductive homogeneous. In fact, a three-dimensional homogeneous Lorentzian manifold is necessarily reductive. This was proved in [7] and also follows independently from the classification obtained by Calvaruso in [2]. Furthermore, the existence of a non-reductive four-dimensional pseudo-Riemannian homogeneous manifolds was proven in [7].

Homogeneous Ricci solitons have been investigated in [8]. A natural generalization of algebraic Ricci solitons on Lie groups to homogeneous spaces is the following [8]:

**Definition 1.2.** Let  $(M = G/H, g)$  be a homogeneous Riemannian manifold. Then  $g$  is called an algebraic Ricci soliton if

$$\text{Ric} = c\text{Id} + pr \circ D \quad (1.4)$$

where  $\text{Ric}$  denotes the Ricci operator of  $m$ ,  $pr : \mathfrak{g} \rightarrow m$ ,  $c$  is a real number, and  $D \in \text{Der}(\mathfrak{g})$ .

Note that, the above definition can be extended to the pseudo-Riemannian case, if changing homogeneous Riemannian manifold into reductive homogeneous pseudo-Riemannian manifold. In [1], we obtained the classification of algebraic Ricci solitons of three-dimensional Lorentzian Lie groups.

In [5], four-dimensional generalized symmetric spaces have been completely classified. They are divided into four classes, named A, B, C and D and the (pseudo-)Riemannian metrics can have any signature. All these spaces are reductive homogeneous. In [3], the Levi Civita connection, the curvature tensor and the Ricci tensor of these spaces are computed. We will use the results of these computations to study the algebraic Ricci solitons of these spaces.

## 2 On generalized symmetric spaces

We start by recalling the definition of generalized symmetric space. Let  $(M, g)$  be a (pseudo-)Riemannian manifold. A *regular s-structure* on  $M$  is a family of isometries  $\{s_p \mid p \in M\}$  of  $(M, g)$  such that

- the mapping  $M \times M \rightarrow M : (p, q) \mapsto s_p(q)$  is smooth,
- $\forall p \in M : p$  is an isolated fixed point of  $s_p$ ,
- $\forall p, q \in M : s_p \circ s_q = s_{s_p(q)} \circ s_p$ .

$s_p$  is called a symmetry centered at  $p$ . The *order* of a regular  $s$ -structure is the smallest integer  $m \geq 2$  such that  $s_p^m = \text{id}_M$  for all  $p \in M$ . If such an integer does not exist, we say that the regular  $s$ -structure has order infinity. A *generalized symmetric space* is a connected, pseudo-Riemannian manifold, carrying at least one regular  $s$ -structure. In particular, a generalized symmetric space is a symmetric space if and only if it admits a regular  $s$ -structure of order

2. The *order* of a generalized symmetric space is the minimum of orders of all possible  $s$ -structures on it. Furthermore, if  $(M, g)$  is a generalized symmetric space then it is homogeneous, that is, the full isometry group  $I(M)$  of  $M$  acts transitively on it, this means that  $(M, g)$  can be identified with  $(G/H, g)$ , where  $G \subset I(M)$  is a subgroup of  $I(M)$  acting transitively on  $M$  and  $H$  is the isotropy group at a fixed point  $o \in M$ .

Generalized symmetric spaces of low dimension have been completely classified. The following Theorem presents a complete classification of four-dimensional pseudo-Riemannian generalized symmetric spaces.

**Theorem 2.1** ([5]). *All non-symmetric, simply connected generalized symmetric spaces  $(M, g)$  of dimension 4 are of order 3 or 4, or infinity. All these spaces are indecomposable, and belong, up to isometry, to the following four types:*

- Type A. *The underlying homogeneous space is  $G/H$ , where*

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*with  $ad - bc = 1$ .  $(M, g)$  is the space  $\mathbb{R}^4(x, y, u, v)$  with the pseudo-Riemannian metric*

$$g = \lambda \left[ (1 + y^2) dx^2 + (1 + x^2) dy^2 - 2xy dx dy \right] / (1 + x^2 + y^2) \quad (2.1) \\ \pm \left[ \left( -x + \sqrt{1 + x^2 + y^2} \right) du^2 + \left( x + \sqrt{1 + x^2 + y^2} \right) dv^2 - 2y^2 du dv \right],$$

*where  $\lambda \neq 0$  is a real constant. The order is  $k = 3$  and possible signature are  $(4, 0), (0, 4), (2, 2)$ .*

- Type B. *The underlying homogeneous space is  $G/H$ , where*

$$G = \begin{pmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^x & 0 & b \\ 0 & 0 & e^y & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*$(M, g)$  is the space  $\mathbb{R}^4(x, y, u, v)$  with the pseudo-Riemannian metric*

$$g = \lambda (dx^2 + dy^2 + dx dy) + e^{-y} (2dx + dy) dv + e^{-x} (dx + 2dy) du \quad (2.2)$$

*where  $\lambda$  is a real constant. The order is  $k = 3$  and the signature is always  $(2, 2)$ .*

- Type C. *The underlying homogeneous space  $G/H$  is the matrix group*

$$G = \begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$(M, g)$  is the space  $\mathbb{R}^4(x, y, z, t)$  with the pseudo-Riemannian metric

$$g = \pm (e^{2t} dx^2 + e^{-2t} dy^2) + dz dt. \quad (2.3)$$

The order is  $k = 3$  and possible signature are  $(1, 3), (3, 1)$ .

- Type D. The underlying homogeneous space is  $G/H$  where

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with  $ad - bc = 1$ .  $(M, g)$  is the space  $\mathbb{R}^4(x, y, u, v)$  with the pseudo-Riemannian metric

$$\begin{aligned} g = & -2 \cosh(2u) \cos(2v) dx dy + \lambda (du^2 - \cosh^2(2u) dv^2) \\ & + (\sinh(2u) - \cosh(2u) \sin(2v)) dx^2 + (\sinh(2u) + \cosh(2u) \sin(2v)) dy^2, \end{aligned} \quad (2.4)$$

where  $\lambda \neq 0$  is a real constant. The order is infinite and the signature is  $(2, 2)$ .

### 3 Algebraic Ricci soliton of spaces of type A with neutral signature

Let  $(M, g)$  be a four-dimensional generalized pseudo-Riemannian symmetric space and denote by  $\nabla$  and  $R$  the Levi-Civita connection and the Riemann curvature tensor of  $M$  respectively. Throughout this paper, we will always use the sign convention

$$R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y].$$

The Ricci tensor of  $(M, g)$  is defined by

$$\varrho(X, Y) = \sum_{k=1}^4 \varepsilon_k g(R(X, e_k)Y, e_k),$$

where  $\{e_1, e_2, e_3, e_4\}$  is a pseudo-orthonormal frame field, with  $g(e_k, e_k) = \varepsilon_k = \pm 1$ . The Ricci operator  $\text{Ric}$  is then given by

$$\varrho(X, Y) = g(\text{Ric}(X), Y).$$

Now, consider a four-dimensional generalized symmetric space  $(M = G/H, g)$  of type A, with signature  $(2, 2)$ . Then, taking into account the results of [5] and [6] the Lie algebra  $\mathfrak{g}$  of the Lie group  $G$  may be decomposed into vector spaces direct sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  where  $\mathfrak{h}$  denotes the Lie algebra of  $H$  and  $\mathfrak{m}$  is a vector space of  $\mathfrak{g}$ .

The Lie algebra  $\mathfrak{g}$  admits a basis  $\{U_1, U_2, U_3, U_4, U_5\}$ , where  $\{U_1, U_2, U_3, U_4\}$  is an orthogonal basis of  $m$  and  $\{U_5\}$  basis of  $\mathfrak{h}$ , such that the Lie bracket  $[\cdot, \cdot]$  on  $\mathfrak{g}$  and the scalar product  $\langle \cdot, \cdot \rangle$  on  $m$  are given, respectively, by

$$\begin{array}{c|ccccc} [\cdot, \cdot] & U_1 & U_2 & U_3 & U_4 & U_5 \\ \hline U_1 & 0 & 0 & -\delta U_1 & \delta U_2 & U_2 \\ U_2 & 0 & 0 & \delta U_2 & \delta U_1 & -U_1 \\ U_3 & \delta U_1 & -\delta U_2 & 0 & -2\delta^2 U_5 & -2U_4 \\ U_4 & -\delta U_2 & -\delta U_1 & 2\delta^2 U_5 & 0 & 2U_3 \\ U_5 & -U_2 & U_1 & 2U_4 & -2U_3 & 0 \end{array} \quad (3.1)$$

where  $\delta > 0$  is a real constant, and

$$\begin{array}{c|cccc} \langle \cdot, \cdot \rangle & U_1 & U_2 & U_3 & U_4 \\ \hline U_1 & 1 & 0 & 0 & 0 \\ U_2 & 0 & 1 & 0 & 0 \\ U_3 & 0 & 0 & -2 & 0 \\ U_4 & 0 & 0 & 0 & -2 \end{array}$$

We start by recalling the following result on the curvature tensor and the Ricci tensor of four-dimensional generalized symmetric spaces of type A (see [3]).

**Lemma 3.1.** *Let  $M$  be a four-dimensional generalized symmetric space of type A, with signature  $(2, 2)$ . Then, there exist a pseudo-orthonormal frame field  $\{e_1 = U_1, e_2 = U_2, e_3 = \frac{1}{\sqrt{2}}U_3, e_4 = \frac{1}{\sqrt{2}}U_4\}$  on  $M$ , with  $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = 1$ . The non-vanishing components of the Levi-Civita connection  $\nabla$  of  $M$  are given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{\delta}{\sqrt{2}} e_3, & \nabla_{e_1} e_2 &= \frac{\delta}{\sqrt{2}} e_4, & \nabla_{e_1} e_3 &= -\frac{\delta}{\sqrt{2}} e_1, & \nabla_{e_1} e_4 &= \frac{\delta}{\sqrt{2}} e_2, \\ \nabla_{e_2} e_1 &= \frac{\delta}{\sqrt{2}} e_4, & \nabla_{e_2} e_2 &= \frac{\delta}{\sqrt{2}} e_3, & \nabla_{e_2} e_3 &= \frac{\delta}{\sqrt{2}} e_2, & \nabla_{e_2} e_4 &= \frac{\delta}{\sqrt{2}} e_1, \end{aligned}$$

The only non-zero components of the Riemann curvature tensor  $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ , with respect to  $\{e_1, e_2, e_3, e_4\}$ , are

$$\begin{aligned} R_{1212} &= -R_{1234} = -\delta^2, \\ R_{1313} &= -R_{1324} = R_{1414} = R_{1423} = R_{2323} = R_{2424} = -\frac{\delta^2}{2}. \end{aligned}$$

Consequently, the non-zero components of the Ricci tensor are given by

$$\varrho(e_3, e_3) = \varrho(e_4, e_4) = -\delta^2.$$

Now, let  $D \in \text{Der}(\mathfrak{g})$  where  $\mathfrak{g}$  is the Lie algebra used in (3.1). Put

$$DU_l = \lambda_l^1 U_1 + \lambda_l^2 U_2 + \lambda_l^3 U_3 + \lambda_l^4 U_4 + \lambda_l^5 U_5 \quad \text{for all } l = 1, \dots, 5.$$

Starting from (3.1), we can write down (1.2) and we get

$$\begin{aligned}
\lambda_3^5 + \delta(2\lambda_1^2 + \lambda_3^4) &= 0, \\
\lambda_3^5 + \delta(2\lambda_2^1 - \lambda_3^4) &= 0, \\
\lambda_4^5 + \delta(\lambda_1^1 - \lambda_2^2) &= 0, \\
\lambda_2^1 - \lambda_1^2 + \lambda_4^3 &= 0, \\
\lambda_1^1 - \lambda_2^2 + \delta\lambda_5^4 &= 0, \\
\lambda_1^2 + \lambda_2^1 + \delta\lambda_5^3 &= 0, \\
\lambda_3^2 + \lambda_4^1 + 2\delta\lambda_5^1 &= 0, \\
\lambda_3^1 - \lambda_4^2 + 2\delta\lambda_5^2 &= 0, \\
2\lambda_4^1 - \lambda_3^2 + \delta\lambda_5^1 &= 0, \\
2\lambda_4^2 + \lambda_3^1 - \delta\lambda_5^2 &= 0, \\
\lambda_4^1 - 2\lambda_3^2 - \delta\lambda_5^1 &= 0, \\
\lambda_4^2 + 2\lambda_3^1 + \delta\lambda_5^2 &= 0, \\
\lambda_3^5 = \delta^2\lambda_5^3, \quad \lambda_4^3 = -\lambda_3^4, \quad \lambda_4^5 = \delta^2\lambda_5^4, \\
\lambda_1^3 = \lambda_1^4 = \lambda_1^5 = \lambda_2^3 = \lambda_2^4 = \lambda_2^5 = \lambda_3^3 = \lambda_4^3 = \lambda_5^3 = 0.
\end{aligned} \tag{3.2}$$

A standard computation proves that all solutions of (3.2) are given by

$$\begin{aligned}
\lambda_2^1 &= -\lambda_1^2 - \delta\lambda_5^3, \quad \lambda_2^2 = \lambda_1^1 + \delta\lambda_5^4, \quad \lambda_3^1 = -\lambda_4^2 = -\delta\lambda_5^2, \\
\lambda_3^2 &= \lambda_4^1 = -\delta\lambda_5^1, \quad \lambda_4^3 = -\lambda_3^4 = 2\lambda_1^2 + \delta\lambda_5^3.
\end{aligned}$$

So, we proved the following.

**Lemma 3.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be the Lie algebra used in (3.1). Then  $D \in \text{Der}(\mathfrak{g})$  if and only if*

$$D = \begin{pmatrix} \lambda_1^1 & -\lambda_1^2 - \delta\lambda_5^3 & -\delta\lambda_5^2 & -\delta\lambda_5^1 & \lambda_5^1 \\ \lambda_1^2 & \lambda_1^1 + \delta\lambda_5^4 & -\delta\lambda_5^1 & \delta\lambda_5^2 & \lambda_5^2 \\ 0 & 0 & 0 & 2\lambda_1^2 + \delta\lambda_5^3 & \lambda_5^3 \\ 0 & 0 & -2\lambda_1^2 - \delta\lambda_5^3 & 0 & \lambda_5^4 \\ 0 & 0 & \delta^2\lambda_5^3 & \delta^2\lambda_5^4 & 0 \end{pmatrix}$$

Using the above lemma, we now prove the following.

**Theorem 3.2.** *Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type A, with signature  $(2, 2)$ . Then,  $M$  is an algebraic Ricci solitons. In particular*

$$pr \circ D = \begin{pmatrix} -\delta^2 & 0 & 0 & 0 \\ 0 & -\delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } c = \delta^2.$$

**Proof.** Using Lemma 3.1 we obtain that the Ricci operator of  $(M = G/H, g)$  is given, with respect to the basis  $\{U_1, U_2, U_3, U_4, U_5\}$ , by

$$\text{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \end{pmatrix}.$$

Hence, the algebraic Ricci soliton condition (1.4) on  $M$  is satisfied if and only if:

$$\begin{aligned} \lambda_1^1 &= -c = -\delta^2, \\ \lambda_1^2 &= \lambda_5^1 = \lambda_5^2 = \lambda_5^3 = \lambda_5^4 = 0. \square \end{aligned}$$

**Remark 3.3.** Algebraic Ricci solitons of spaces of type A, the Riemannian case, is obtained by the following. If we change  $U_3$  and  $U_4$  of pseudo-Riemannian case to  $\frac{1}{\delta}U_3$  and  $\frac{1}{\delta}U_4$  and put  $\rho = -\delta^2$ , we obtain the Riemannian case of the Lie bracket. So, it is easy to check that this case has a soliton.

## 4 Algebraic Ricci soliton of spaces of type B

Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type B, with signature  $(2, 2)$ . Then,  $\mathfrak{g} = \mathfrak{h} \oplus m$  and  $\{U_1, U_2, U_3, U_4\}$  and  $\{U_5\}$  are respectively a basis of  $m$  and  $\mathfrak{h}$ , such that the Lie bracket  $[\cdot, \cdot]$  on  $\mathfrak{g}$  and the scalar product  $\langle \cdot, \cdot \rangle$  on  $m$  are given, respectively, by

$[\cdot, \cdot]$	$U_1$	$U_2$	$U_3$	$U_4$	$U_5$	
$U_1$	0	0	$-U_1$	$\varepsilon U_5 + U_2$	0	
$U_2$	0	0	$-\varepsilon U_5 + U_2$	$U_1$	0	
$U_3$	$U_1$	$\varepsilon U_5 - U_2$	0	0	$2U_2$	
$U_4$	$-\varepsilon U_5 - U_2$	$-U_1$	0	0	$-2U_1$	
$U_5$	0	0	$-2U_2$	$2U_1$	0	(4.1)

where  $\varepsilon = \pm 1$ , and

$\langle \cdot, \cdot \rangle$	$U_1$	$U_2$	$U_3$	$U_4$
$U_1$	0	0	-1	0
$U_2$	0	0	0	-1
$U_3$	-1	0	$2\lambda$	0
$U_4$	0	-1	0	$2\lambda$

The following result was proven in [3].



**Lemma 4.1.** *Let  $M$  be a four-dimensional generalized symmetric space of type  $B$ , with signature  $(2, 2)$ . Then, there exist a pseudo-orthonormal frame field*

$$\begin{aligned} e_1 &= \left(\lambda - \frac{1}{2}\right) U_1 + U_2, & e_2 &= \left(\lambda - \frac{1}{2}\right) U_3 + U_4, \\ e_3 &= \left(\lambda + \frac{1}{2}\right) U_1 + U_2, & e_4 &= \left(\lambda + \frac{1}{2}\right) U_3 + U_4, \end{aligned}$$

on  $M$ , with  $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = 1$ . The Levi-Civita connection  $\nabla$  of  $M$  is determined by

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3, & \nabla_{e_2} e_1 &= e_4, & \nabla_{e_3} e_1 &= -e_3, & \nabla_{e_4} e_1 &= e_4, \\ \nabla_{e_1} e_2 &= e_4, & \nabla_{e_2} e_2 &= e_3, & \nabla_{e_3} e_2 &= e_4, & \nabla_{e_4} e_2 &= e_3, \\ \nabla_{e_1} e_3 &= -e_1, & \nabla_{e_2} e_3 &= e_2, & \nabla_{e_3} e_3 &= -e_1, & \nabla_{e_4} e_3 &= e_2, \\ \nabla_{e_1} e_4 &= e_2, & \nabla_{e_2} e_4 &= e_1, & \nabla_{e_3} e_4 &= e_2, & \nabla_{e_4} e_4 &= e_1. \end{aligned}$$

The only non-zero components of the Riemann curvature tensor  $R$ , with respect to  $\{e_1, e_2, e_3, e_4\}$ , are

$$R_{1212} = R_{1214} = -R_{1223} = -R_{1234} = -R_{1434} = R_{2334} = -R_{3434} = -2.$$

Consequently, the non-zero components of the Ricci tensor are given by

$$\begin{aligned} \varrho(e_1, e_1) &= \varrho(e_2, e_2) = \varrho(e_3, e_3) = \varrho(e_4, e_4) = -2, \\ \varrho(e_1, e_3) &= \varrho(e_2, e_4) = -4. \end{aligned}$$

Next, let  $D \in \text{Der}(\mathfrak{g})$  where  $\mathfrak{g}$  is the Lie algebra used in (4.1) and put

$$DU_l = \lambda_l^1 U_1 + \lambda_l^2 U_2 + \lambda_l^3 U_3 + \lambda_l^4 U_4 + \lambda_l^5 U_5 \quad \text{for all } l = 1, \dots, 5.$$

Using (4.1), we prove that (1.2) is satisfied if and only if

$$\begin{aligned} \lambda_3^4 + 2(\lambda_1^2 - \lambda_1^5) &= 0, \\ \lambda_1^5 + \varepsilon(\lambda_3^4 - \lambda_1^2) &= 0, \\ \lambda_1^2 - \lambda_4^3 + 2\lambda_1^5 - \varepsilon\lambda_5^1 - \lambda_2^1 &= 0, \\ \lambda_1^1 + \lambda_4^4 - \varepsilon\lambda_5^2 - \lambda_2^2 &= 0, \\ \lambda_2^5 - \varepsilon(\lambda_1^1 + \lambda_4^4 - \lambda_5^5) &= 0, \\ \lambda_3^4 - 2\lambda_2^1 + \varepsilon\lambda_5^1 &= 0, \\ \lambda_2^5 + \varepsilon(\lambda_2^2 - \lambda_5^5) &= 0, \\ \lambda_1^1 - \lambda_2^2 - \lambda_4^4 - 2\lambda_2^5 &= 0, \\ \lambda_1^2 - \lambda_2^1 - \lambda_4^3 &= 0, \\ \lambda_1^5 + \varepsilon(\lambda_4^3 - \lambda_2^1) &= 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned}
\lambda_3^2 + \lambda_4^1 + 2\lambda_3^5 &= 0, \\
\lambda_3^1 - \lambda_4^2 + 2\lambda_4^5 &= 0, \\
\lambda_5^1 - 2(\lambda_2^1 + \lambda_3^4) &= 0, \\
\lambda_5^2 + 2(\lambda_2^2 - \lambda_5^5) &= 0, \\
\lambda_5^2 + 2(-\lambda_1^1 + \lambda_4^4 + \lambda_5^5) &= 0, \\
\lambda_5^1 - 2(\lambda_1^2 + \lambda_4^3) &= 0, \\
\lambda_4^2 = -\lambda_3^1, \quad \lambda_5^1 = 2\varepsilon\lambda_1^5, \quad \lambda_5^2 = 2\varepsilon\lambda_2^5, \\
\lambda_1^3 = \lambda_1^4 = \lambda_2^3 = \lambda_2^4 = \lambda_3^3 = \lambda_3^5 = \lambda_4^4 = 0.
\end{aligned}$$

So, we need to consider two cases:

- If  $\varepsilon = 1$ . In this case, we prove that all solutions of (4.2) are given by

$$\begin{aligned}
\lambda_1^5 = \lambda_2^1 = \lambda_1^2, \quad \lambda_4^1 = -\lambda_3^2 - 2\lambda_3^5, \quad \lambda_4^2 = \lambda_4^5 = -\lambda_3^1, \\
\lambda_2^2 = \lambda_1^1 - 2\lambda_2^5, \quad \lambda_5^1 = 2\lambda_1^2, \quad \lambda_5^2 = 2\lambda_2^5, \quad \lambda_5^5 = \lambda_1^1 - \lambda_2^5, \\
\lambda_3^4 = \lambda_4^3 = \lambda_4^4 = 0.
\end{aligned}$$

- If  $\varepsilon = -1$ , all solutions of (4.2) are given by

$$\begin{aligned}
\lambda_4^1 = -\lambda_3^2 - 2\lambda_3^5, \quad \lambda_4^2 = \lambda_4^5 = -\lambda_3^1, \quad \lambda_5^5 = \lambda_2^2 = \lambda_1^1, \\
\lambda_1^2 = \lambda_1^5 = \lambda_2^1 = \lambda_2^5 = \lambda_3^4 = \lambda_4^3 = \lambda_4^4 = \lambda_5^1 = \lambda_5^2 = 0.
\end{aligned}$$

Therefore, we proved the following.

**Lemma 4.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus m$  be the Lie algebra used in (4.1). Then  $D \in \text{Der}(\mathfrak{g})$  if and only if*

- $\varepsilon = 1$

$$D = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_3^1 & -\lambda_3^2 - 2\lambda_3^5 & 2\lambda_1^2 \\ \lambda_1^2 & \lambda_1^1 - 2\lambda_2^5 & \lambda_3^2 & -\lambda_3^1 & 2\lambda_2^5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda_1^2 & \lambda_2^5 & \lambda_3^5 & -\lambda_3^1 & \lambda_1^1 - \lambda_2^5 \end{pmatrix}$$

- $\varepsilon = -1$

$$D = \begin{pmatrix} \lambda_1^1 & 0 & \lambda_3^1 & -\lambda_3^2 - 2\lambda_3^5 & 0 \\ 0 & \lambda_1^1 & \lambda_3^2 & -\lambda_3^1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^5 & \lambda_3^5 & -\lambda_3^1 & \lambda_1^1 \end{pmatrix}.$$

Using the above lemma, we now prove the following.

**Theorem 4.2.** *Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type B. Then,  $M$  is not an algebraic Ricci soliton.*

**Proof.** Using Lemma 4.1 we obtain that the Ricci operator of  $(M = G/H, g)$  is given, with respect to the basis  $\{U_1, U_2, U_3, U_4, U_5\}$ , by

$$\text{Ric} = \begin{pmatrix} -4\lambda & 0 & 4\lambda^2 + 3 & 0 \\ 0 & -4\lambda & 0 & 4\lambda^2 + 3 \\ -4 & 0 & 4\lambda & 0 \\ 0 & -4 & 0 & 4\lambda \end{pmatrix}.$$

Hence it follows, from the above lemma, that the algebraic Ricci soliton condition (1.4) on  $M$  does not occur.  $\square$

## 5 Algebraic Ricci soliton of spaces of type C

Let  $(M = G, g)$  be a four-dimensional generalized symmetric space of type C. Without loss of generality, we assume that the signature is  $(3, 1)$ . The Lie algebra  $\mathfrak{g}$  admits a basis  $\{U_1, U_2, U_3, U_4\}$ , such that the Lie bracket  $[\cdot, \cdot]$  and the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  are given, respectively, by

$$\begin{array}{c|cccc} [\cdot, \cdot] & U_1 & U_2 & U_3 & U_4 \\ \hline U_1 & 0 & 0 & 0 & -U_1 \\ U_2 & 0 & 0 & 0 & U_2 \\ U_3 & 0 & 0 & 0 & 0 \\ U_4 & U_1 & -U_2 & 0 & 0 \end{array} \quad (5.1)$$

and

$$\begin{array}{c|cccc} \langle \cdot, \cdot \rangle & U_1 & U_2 & U_3 & U_4 \\ \hline U_1 & 1 & 0 & 0 & 0 \\ U_2 & 0 & 1 & 0 & 0 \\ U_3 & 0 & 0 & 0 & 1/2 \\ U_4 & 0 & 0 & 1/2 & 0 \end{array}$$

The following result was proven in [3].

**Lemma 5.1.** *Let  $M$  be a four-dimensional generalized symmetric space of type C, with signature  $(3, 1)$ . Then, there exist a pseudo-orthonormal frame field*

$$e_1 = U_1, \quad e_2 = U_2, \quad e_3 = U_3 + U_4, \quad e_4 = U_3 - U_4,$$

*on  $M$ , with  $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = 1$ . The non-vanishing components of the Levi-Civita connection  $\nabla$  of  $M$  are given by*

$$\nabla_{e_1} e_1 = -\nabla_{e_2} e_2 = e_3 + e_4, \quad \nabla_{e_1} e_4 = -\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_3 = -\nabla_{e_2} e_4 = e_2.$$

The non-zero components of the Riemann curvature tensor  $R$ , with respect to  $\{e_1, e_2, e_3, e_4\}$ , are

$$R_{1313} = -R_{1314} = R_{1414} = R_{2323} = -R_{2324} = R_{2424} = -1.$$

Consequently, the only non-zero components of the Ricci tensor are given by

$$\varrho(e_3, e_3) = \varrho(e_4, e_4) = -\varrho(e_3, e_4) = -2.$$

Next, put  $DU_l = \lambda_l^1 U_1 + \lambda_l^2 U_2 + \lambda_l^3 U_3 + \lambda_l^4 U_4$  for all  $l = 1, \dots, 4$ , where  $\{U_1, U_2, U_3, U_4\}$  is the basis used in (5.1). Standard computations proves that  $D \in \text{Der}(\mathfrak{g})$  if and only if

$$\lambda_1^2 = \lambda_1^3 = \lambda_1^4 = \lambda_2^1 = \lambda_2^3 = \lambda_2^4 = \lambda_3^1 = \lambda_3^2 = \lambda_3^4 = \lambda_4^1 = 0.$$

So, we deduce the following.

**Lemma 5.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be the Lie algebra used in (5.1). Then  $D \in \text{Der}(\mathfrak{g})$  if and only if*

$$D = \begin{pmatrix} \lambda_1^1 & 0 & 0 & \lambda_4^1 \\ 0 & \lambda_2^2 & 0 & \lambda_4^2 \\ 0 & 0 & \lambda_3^3 & \lambda_4^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can now prove the following.

**Theorem 5.2.** *Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type C. Then,  $M$  is an algebraic Ricci solitons. In particular*

$$D = \text{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } c = 0.$$

**Proof.** Using Lemma 5.1 we write down the Ricci operator of  $(M = G/H, g)$ , with respect to the basis  $\{U_1, U_2, U_3, U_4\}$ , getting

$$\text{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, using the above lemma, we obtain that the algebraic Ricci soliton condition (1.4) on  $M$  is satisfied if and only if

$$\lambda_1^1 = \lambda_2^2 = \lambda_3^3 = \lambda_4^4 = \lambda_1^2 = \lambda_2^1 = c = 0 \text{ and } \lambda_4^3 = -4. \square$$

## 6 Algebraic Ricci soliton of spaces of type D

Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type D, with signature  $(2, 2)$ . The Lie algebra  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  of the Lie group  $G$  admits a basis  $\{U_1, U_2, U_3, U_4, U_5\}$ , with  $\{U_1, U_2, U_3, U_4\}$  and  $\{U_5\}$  are respectively a basis of  $\mathfrak{m}$  and of  $\mathfrak{h}$ , such that

$$\begin{array}{c|ccccc}
 [\cdot, \cdot] & U_1 & U_2 & U_3 & U_4 & U_5 \\
 \hline
 U_1 & 0 & 0 & 0 & -U_2 & U_1 \\
 U_2 & 0 & 0 & -U_1 & 0 & -U_2 \\
 U_3 & 0 & U_1 & 0 & -U_5 & 2U_3 \\
 U_4 & U_2 & 0 & U_5 & 0 & -2U_4 \\
 U_5 & -U_1 & U_2 & -2U_3 & 2U_4 & 0
 \end{array} \tag{6.1}$$

and

$$\begin{array}{c|cccc}
 \langle \cdot, \cdot \rangle & U_1 & U_2 & U_3 & U_4 \\
 \hline
 U_1 & 0 & 1 & 0 & 0 \\
 U_2 & 1 & 0 & 0 & 0 \\
 U_3 & 0 & 0 & 0 & \lambda \\
 U_4 & 0 & 0 & \lambda & 0
 \end{array}$$

with  $\lambda \neq 0$  is a real constant.

The following result was proven in [3].

**Lemma 6.1.** *Let  $M$  be a four-dimensional generalized symmetric space of type D, with signature  $(2, 2)$ . Then, there exist a pseudo-orthonormal frame field*

$$\begin{aligned}
 e_1 &= \frac{1}{\sqrt{2}}(U_1 + U_2), & e_2 &= \frac{1}{\sqrt{2|\lambda|}}(U_3 + \varepsilon U_4), \\
 e_3 &= \frac{1}{\sqrt{2}}(U_1 - U_2), & e_4 &= \frac{1}{\sqrt{2|\lambda|}}(U_3 - \varepsilon U_4),
 \end{aligned}$$

on  $M$ , with  $\varepsilon = \pm 1$  and  $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = 1$ . The

non-vanishing components of the Levi-Civita connection  $\nabla$  of  $M$  are given by

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon + 1) e_2 + (\varepsilon - 1) e_4), \\
\nabla_{e_1} e_2 &= -\frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon + 1) e_1 - (\varepsilon - 1) e_3), \\
\nabla_{e_1} e_3 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon - 1) e_2 + (\varepsilon + 1) e_4), \\
\nabla_{e_1} e_4 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon - 1) e_1 - (\varepsilon + 1) e_3), \\
\nabla_{e_3} e_1 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon - 1) e_2 + (\varepsilon + 1) e_4), \\
\nabla_{e_3} e_2 &= \frac{1}{2\sqrt{2|\lambda|}} ((1 - \varepsilon) e_1 + (\varepsilon + 1) e_3), \\
\nabla_{e_3} e_3 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon + 1) e_2 + (\varepsilon - 1) e_4), \\
\nabla_{e_3} e_4 &= \frac{1}{2\sqrt{2|\lambda|}} ((\varepsilon + 1) e_1 - (\varepsilon - 1) e_3).
\end{aligned}$$

The non-zero components of the Riemann curvature tensor, with respect to  $\{e_1, e_2, e_3, e_4\}$ , are

$$\begin{aligned}
R_{1212} &= -R_{1234} = -R_{1414} = -R_{1423} = -R_{2323} = R_{3434} = -\frac{1}{2\lambda}, \\
R_{1313} &= -R_{1324} = -\frac{1}{\lambda}.
\end{aligned}$$

Consequently, the only non-zero components of the Ricci tensor are given by

$$\varrho(e_2, e_2) = -\varrho(e_4, e_4) = -\frac{1}{\lambda}.$$

Now, let  $D \in \text{Der}(\mathfrak{g})$  where  $\mathfrak{g}$  is the Lie algebra used in (6.1). Put

$$DU_l = \lambda_l^1 U_1 + \lambda_l^2 U_2 + \lambda_l^3 U_3 + \lambda_l^4 U_4 + \lambda_l^5 U_5 \quad \text{for all } l = 1, \dots, 5.$$

Starting from (6.1), we can write down (1.2) and we get

$$\begin{aligned}
\lambda_3^3 &= \lambda_1^1 - \lambda_2^2, \quad \lambda_3^5 = \lambda_1^2, \quad \lambda_4^4 = -\lambda_3^3, \quad \lambda_4^5 = -\lambda_2^1, \\
\lambda_5^1 &= -\lambda_4^2, \quad \lambda_5^2 = \lambda_3^1, \quad \lambda_5^3 = 2\lambda_2^1, \quad \lambda_5^4 = -2\lambda_1^2, \\
\lambda_1^3 &= \lambda_1^4 = \lambda_1^5 = \lambda_2^3 = \lambda_2^4 = \lambda_2^5 = \lambda_3^2 = \lambda_3^4 = \lambda_4^1 = \lambda_4^3 = \lambda_5^5 = 0.
\end{aligned}$$

We deduce the following.

**Lemma 6.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus m$  be the Lie algebra used in (6.1). Then  $D \in \text{Der}(\mathfrak{g})$  if and only if*

$$D = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & 0 & -\lambda_4^2 \\ \lambda_1^2 & \lambda_2^2 & 0 & \lambda_4^2 & \lambda_3^1 \\ 0 & 0 & \lambda_1^1 - \lambda_2^2 & 0 & 2\lambda_2^1 \\ 0 & 0 & 0 & \lambda_2^2 - \lambda_1^1 & -2\lambda_1^2 \\ 0 & 0 & \lambda_1^2 & -\lambda_2^1 & 0 \end{pmatrix}.$$

Using the above lemma, we prove the following.

**Theorem 6.2.** *Let  $(M = G/H, g)$  be a four-dimensional generalized symmetric space of type D. Then,  $M$  is an algebraic Ricci solitons. In particular*

$$pr \circ D = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } c = -\frac{1}{\lambda}.$$

**Proof.** Using Lemma 6.1 we write down the Ricci operator of  $(M = G/H, g)$ , with respect to the basis  $\{U_1, U_2, U_3, U_4\}$ , getting

$$\text{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & -\frac{1}{\lambda} \end{pmatrix}.$$

Using the above lemma, we obtain that the algebraic Ricci soliton condition (1.4) on  $M$  is satisfied if and only if

$$\lambda_1^1 = \lambda_2^2 = -c = \frac{1}{\lambda} \text{ and } \lambda_1^2 = \lambda_2^1 = \lambda_3^1 = \lambda_3^3 = \lambda_4^2 = 0. \square$$

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